Measles Epidemic: the SIR model

Yang Wu[a,1]

[a]Kenyon College

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The following report is written as requested by the Center for Health Studies (CHS). A measles outbreak has spread to an isolated island with 50,000 inhabitants. The situation is currently being monitored by health professionals and academics alike. I will be joining in their efforts to help alleviate the crisis. This paper outlines a epidemiological model, known as the SIR model, attributed to Kermack and Mckendrick, which I will be using to provide estimations for monitoring the situation and guiding policy response.

Measles epidemic | SIR model | ...

The population of the island has been divided into three sub-population groups: currently healthy individuals who are susceptible to the disease (S), infected individuals (I) who are capable of transmitting the disease to others, and recovered individuals who are subsequently immune to further complications or relapses (R).

Model

1. Equations

Members of the population move through the sub-population groups according to the following flowchart:

Fig. 1. The flowchart above captures the entire island population, ignoring births, mortality, and immigration & emigration

The arrows in Figure 1 indicate that the three sub-populations are constantly changing, expanding or declining as the epidemic prolongs. The number of people in each group can be described numerically by three functions: S,I, and R. All these are functions of the time, t, and they change according to a system of differential equations:

$$S' = -\alpha SI \quad [1]$$

Equation 1 describes the rate at which the susceptible population becomes infected as a result of contact with the infected for some positive constant, α, also known as the transmission rate. The susceptible population is declining while the epidemic lasts, and thus the rate of change is negative.

$$R' = \beta I \quad [2]$$

Equation 2 describes the rate at which the recovered population increases as infected individuals recover for some positive constant, β, also called the recovery rate.

$$I' = \alpha SI - \beta I \quad [3]$$

Finally, equation 3 describes the rate at which the infected population changes as the susceptible become infected and the currently infected individual recover.

Analysis

2. Parameters, Estimations

According to reports from the front-line, as of Wednesday, Jan 29, there are 45,400 susceptible individuals, 2,100 infected individuals, and 2500 recovered individuals. Dr.Lois has estimated that measles last about 14 days ($\beta = \frac{1}{14}$)and the transmission rate is 0.00001. Using the computational capabilities of Maple, Iestimate the population in each sub-population group on the two specific dates as requested by the doctor. The results are tabulated below:

Table 1. Euler estimation results

	Sub-population groups:		
	Susceptible (S)	Infected (I)	Recovered (R)
Sunday, Jan 26	47,120.83635	644.9191	2,234.2447
Saturday, Feb 8	13,062.2806	25,546.8698	11,390.8496

*The total population S+I+R always equals to 50,000; it is constant since $\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = -\alpha SI + (\alpha SI - \beta I) + \beta I = 0$
*The values in the table are obtained using a step size of 0.01; more accurate estimations could be obtained by decreasing the step-size, but I believe that, in the context of this analysis, a step size of 0.01 is sufficient

3. Predictions and Graphical Analysis

1. Using the Euler method(a step-size of 0.01), I estimate that the epidemic will die out in ≈ 170 days. I quality the phrase "die out" by assuming that I=0. In other words, I find the value of t such that I≈0. The Euler estimation outputs are tabulated below:

Table 2. Euler estimation results for populations when the epidemic dies out

	Sub-population groups:		
	Susceptible (S)	Infected (I)	Recovered (R)
Tuesday, July 14	58.9061	0.4952	49,940.5987

*The values in the table are obtained using a step size of 0.01; more accurate estimations could be obtained by decreasing the step-size, but I believe that, in the context of this analysis, a step size of 0.01 is sufficient

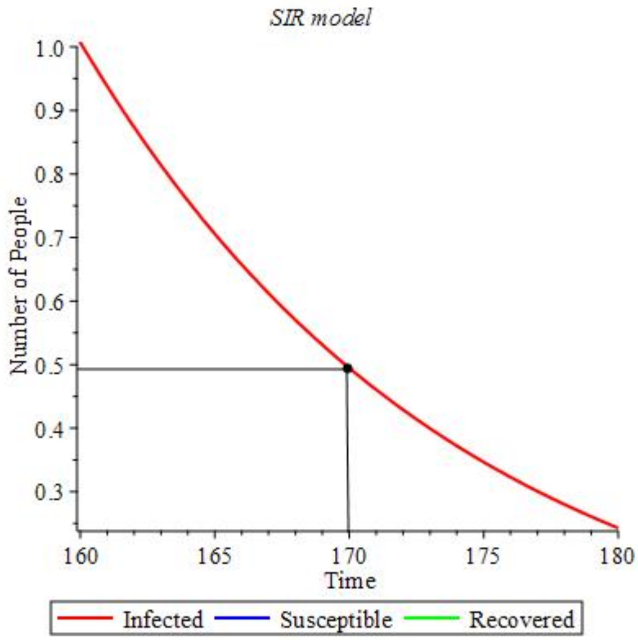

Fig. 2. is zoomed in to better capture the behavior of the curve around t=170, which is approximately when the epidemic dies out

As can be seen from Figure 2, the graphical analysis confirms our prediction that the epidemic will die out in ≈ 170 days. The number of infected is ≈ 0.4952, which is effectively zero.

2. Using the model, I are also able to predict whether there would be an outbreak on the island. Our work are as follows:

Given the rate of change of the infected population 3,

$$I' = \alpha SI - \beta I$$
$$= I\alpha(S - \frac{\beta}{\alpha})$$

Factor out I and α. Notice that, because α and I are both positive, the sign of I' is dependent on the term $(S - \frac{\beta}{\alpha})$, therefore:

$$\text{If S} > \frac{\beta}{\alpha}, \text{then I'} \in \mathbb{R}_{>0}$$
$$\text{If S} < \frac{\beta}{\alpha}, \text{then I'} \in \mathbb{R}_{<0}$$

I conclude that I', the rate of change of the infected population, must equal to zero when S=$\frac{\beta}{\alpha}$; this concept is known as the disease threshold, where, unless the population of susceptible is larger than the ratio of the two parameters, α and β, the number of infected will decrease. Given the parameters of our model, I find:

$$\frac{\beta}{\alpha} = \frac{\frac{1}{14}}{0.00001} \approx 7,142.8571$$

As of Wednesday, Jan 29, the population of susceptible individuals is 45,400, which clears the threshold by the thousands. I conclude that the island will be subject to an measles epidemic.

2. In addition, I am able to predict the infected population at the peak of the epidemic. Our work are as follows:

I', the rate of change of the infected population, involves two unknown functions, S and I. I use the definition of the chain rule to manipulate the differential equations so that I may have a single differential equation that involves a single unknown function. I entertain I as a function of S, and its rate of change is given by the following differential equation:

$$I' = \frac{dI}{dS}$$

According to the chain rule:

$$\frac{dI}{dS} = \frac{dI}{dt} * \frac{dt}{dS}$$
$$= \frac{\frac{dI}{dt}}{\frac{dS}{dt}}$$
$$= \frac{\alpha SI - \beta I}{-\alpha SI}$$

Factor out I and simplify:

$$= \frac{I(\alpha S - \beta)}{I(-\alpha S)}$$
$$= \frac{\alpha S - \beta}{-\alpha S}$$

I now have the following equation for I', involving one unknown function, S:

$$\frac{dI}{dS} = \frac{\alpha S - \beta}{-\alpha S}$$

Taking the indefinite integral:

$$I = \frac{\beta}{\alpha} ln \mid \text{S} \mid - S + C, \text{where C is a constant of integration}$$

Detailed steps for the integration process is included in the Appendix 4.Using the initial conditions and the parameters to solve for C:

$$C = I(0) - \frac{\beta}{\alpha} ln \mid \text{S}(0) \mid - S(0)$$
$$= 2,100 - \frac{\frac{1}{14}}{0.00001} ln \mid 45,400 \mid - 45,400$$
$$\approx -290,094.7670$$

I have the following function, I:

$$I = \frac{\beta}{\alpha} ln \mid \text{S} \mid - S - 290,094.7670$$

At the peak of the epidemic, the infected population has reached its maximum value, where no increase is possible and the population has not yet begin to decline. As this juncture, I' must be equal to zero. I know from our earlier work:

$$\text{When I'} = 0, \text{it must be true that S} = \frac{\beta}{\alpha},$$

That is, S equals the disease threshold when I' is zero. We found the disease threshold to be ≈7,142.8571; plugging in the disease threshold for S in the function I:

$$I = \frac{\beta}{\alpha} ln \mid 7,142.8571 \mid - (7,142.8571) + 290,094.7670$$
$$\approx 27,147.1482$$

So, we predict that about 27,147.1482 individuals would be infected at the peak of the epidemic. Knowing these values for S and I and using maple's looping capabilities, we find that the epidemic peaks in about 12.26 days starting from Wednesday, January 29. The Euler estimation outputs are tabulated below:

Table 3. Euler estimation results for peak populations

	Sub-population groups:		
	Susceptible (S)	Infected (I)	Recovered (R)
Monday, February 10	7,148.7315	27,160.4877	15,690.7807

*The values in the table are obtained using a step size of 0.01; more accurate estimations could be obtained by decreasing the step-size, but we believe that, in the context of this analysis, a step size of 0.01 is sufficient

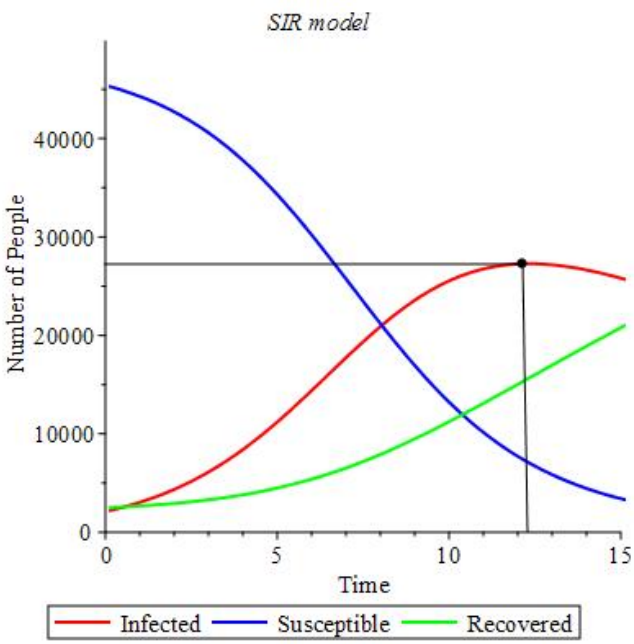

Fig. 3. depicts the behaviors of I, S, and R, around t=12.26, which is approximately when the epidemic peaks

As can be seen from Figure 3, the graphical analysis confirms our prediction that the epidemic will peak in ≈ 12.26 days. Peak infected population is ≈ 27,160.4877, which is close to the exact value I solved for analytically.

4. Inference

What does the model communicate? Given the urgency of the situation, it is pivotal for policy makers act quickly. Recent reports from the front-line have indicated that a quarantine has been put into effect. According to Dr.Lois' estimation, a quarantine effectively reduces the transmission rate, α, by half. I find the new transmission rate to be an improved but nevertheless insufficient effort in combating the outbreak:

$$\begin{aligned} S &= \frac{\beta}{\text{New}\alpha} \\ &= \frac{\frac{1}{14}}{\frac{0.00001}{2}} \\ &\approx 14,285.7143 \end{aligned}$$

The current population of susceptible, 45,400 people, is still too large for an infected population to never grow, fostering into an epidemic. Furthermore, a graphical analysis reveals that the time at which the epidemic would die out is delayed under the quarantine:

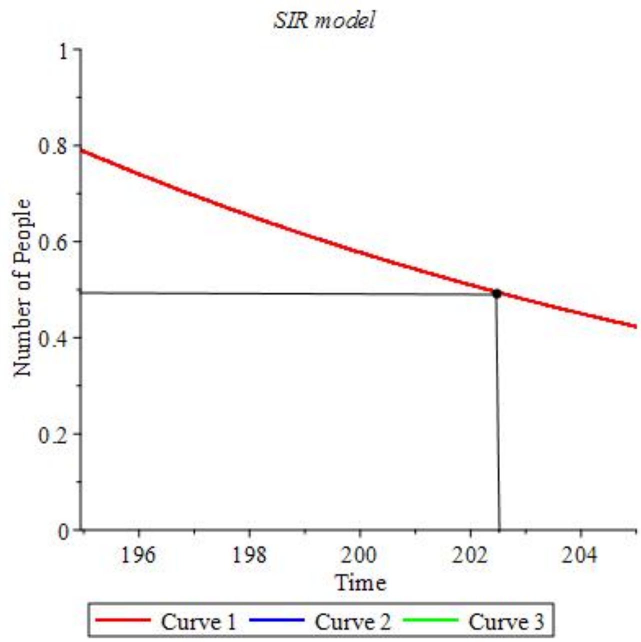

Fig. 4. is zoomed in to better capture the behavior of the curve around t=202.23, which is approximately when the epidemic dies out under the quarantine

As can be seen in figure 4, under the quarantine, the infected population is predicted to become effectively zero in ≈ 202.23 days, which is a delay compared to that of without it, ≈ 170 days. The Euler estimation outputs for the populations when the epidemic dies out under the quarantine are tabulated below:

Table 4. Euler estimation results for populations when epidemic dies out (Under Quarantine)

	Sub-population groups:		
	Susceptible (S)	Infected (I)	Recovered (R)
Saturday, August 15	1,858.4766	0.4999	48,141.0240

*The values in the table are obtained using a step size of 0.01; more accurate estimations could be obtained by decreasing the step-size, but I believe that, in the context of this analysis, a step size of 0.01 is sufficient

From Table 4, I see that the susceptible population under the quarantine is 1,858.4766 when the epidemic dies out; this value is larger than the susceptible population without the quarantine, in Table 2, 58.9061. Our interpretation is that, despite that fact that it takes longer for the epidemic to die out under the quarantine, the implementation of it is associated with a larger susceptible population,i.e. those who never get sick, at the end of the epidemic. In other words, fewer people would become infected under the quarantine. Nevertheless, using the concept of the threshold, I conclude that the transmission rate ought to be reduced further:

$$\text{current susceptible population} > \frac{\beta}{\alpha}$$

The infected population grows. I would like to find a transmission rate, α, such that:

$$\begin{aligned}\text{current susceptible population} &= \frac{\beta}{\text{New}\alpha}\\ \text{New}\alpha &= \frac{\beta}{\text{current susceptible population}}\\ \text{New}\alpha &= \frac{\frac{1}{14}}{45,400}\\ &\approx 1.5733E{-6}\end{aligned}$$

Compared this value to the original transmission rate, 0.00001:

$$\begin{aligned}\frac{\text{original}\alpha}{\text{New}\alpha} &= \frac{0.00001}{1.5733E{-6}}\\ &= 6.356\end{aligned}$$

I interpret this number as the factor by which the transmission rate must be reduced, i.e, in order to alleviate the measles epidemic on the island, the transmission rate must be reduced to 1/7 of what it was without the quarantine. Given the urgency of the situation, a stricter quarantine and, perhaps, other public health programs are strongly advisable. While vaccines, which reduce the population of susceptible, are a viable option, it may be that we are too far into the epidemic for vaccines to lead to any remarkable turnaround. Lastly, with antibiotics, which increases the recovery rate, β, in our model, still months away, it may be advisable to scale up production of medical supplies to support intermediate or temporary treatments. In times of health epidemic, any given economy must allocate its resources toward preventing a social breakdown.

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Appendix

We have the following equation for I':

$$\frac{dI}{dS} = \frac{\alpha S - \beta}{-\alpha S}$$

Technically, the variables have already been separated, so we simply write out the implicit 1 on the left side of the equation:

$$(1)\frac{dI}{dS} = \frac{\alpha S - \beta}{-\alpha S}$$

Take the integral with respect to S:

$$\int (1)\frac{dI}{dS}dS = \int \frac{\alpha S - \beta}{-\alpha S}dS$$
$$\int (1)dI = \int \frac{\alpha S - \beta}{-\alpha S}dS$$

Now we focus on the left side of the equation, integrate with respect to I:

$$\int (1)dI = I + C_1, \text{where } C_1 \text{ is a constant of integration}$$

On the right side of the equation we have:

$$\int \frac{\alpha S - \beta}{-\alpha S}dS$$

Apply the constant multiple rule

$$\frac{1}{-\alpha}\int \frac{\alpha S - \beta}{S}dS$$

Apply the fraction rule and simplify:

$$\frac{1}{-\alpha}\int \frac{\alpha S}{S} - \frac{\beta}{S}dS$$
$$\frac{1}{-\alpha}\int \alpha - \frac{\beta}{S}dS$$

Apply the differences rule:

$$\frac{1}{-\alpha}\int \alpha dS - \int \frac{\beta}{S}dS$$

Integrate with respect to S:

$$(\frac{1}{-\alpha})[(\alpha S + C_1) - (\beta \ln|S| + C_2)]$$
$$(\frac{1}{-\alpha})(\alpha S + C_1 - \beta \ln|S| - C_2)$$

Combine the constant terms C_1 and C_2:

$$(\frac{1}{-\alpha})(\alpha S - \beta \ln|S| + C), \text{where C is a constant of integration}$$

Distribute and Simplify

$$\frac{\alpha S}{-\alpha} + \frac{\beta \ln|S|}{\alpha} + \frac{1}{-\alpha}C, \text{where C is a constant of integration}$$
$$\frac{\alpha S}{-\alpha} + \frac{\beta \ln|S|}{\alpha} + C_2, \text{where } \frac{1}{-\alpha}C = C_2 \text{ as a constant of integration}$$
$$-S + \frac{\beta \ln|S|}{\alpha} + C_2, \text{where } C_2 \text{ is a constant of integration}$$

Set the left side of the equation equal to the right:

$$I + C_1 = -S + \frac{\beta \ln|S|}{\alpha} + C_2$$

Combine the constant terms C_1 and C_2 and re-arrange:

$$I = \frac{\beta \ln|S|}{\alpha} - S + C, \text{where } C = C_1 + C_2 \text{ as a constant of integration}$$